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Introduction to Schramm-Loewner Evolution and its Application to Critical Systems

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In this short review we look at recent advances in Schramm-Loewner Evolution (SLE) theory and its application to critical phenomena. The application of SLE goes beyond critical systems to other time dependent, scale invariant phenomena such as turbulence, sand-piles and watersheds. Through the use of SLE, the evolution of conformally invariant paths on the complex plane can be followed; hence a geometrical interpretation is acquired for domain walls in critical phenomena. Also advances made on rigorous mathematical proofs in particular for the Ising and percolation models are noteworthy, giving rise to rigorous results in critical phenomena. On the other hand; application of numerical techniques to SLE for systems far from equilibrium such as surface growth has yielded interesting new results. For example it has yielded results regarding the universality class of certain models which have all been thought to belong to the class of Kardar-Parisi-Zhang model. In this short review we will present some of these results.

Keywords: Critical phenomena, Stochastic processes, Conformal invariance

INTRODUCTION

The theory of Schramm-Loewner evolution (SLE) is a topic which applies complex analysis to critical phenomena. SLEs are random, non-intersecting, planar curves which are characterized by certain conformal invariance properties. The theory of SLE was created by Oded Schramm [**1**] and since then it has been a very active field in Mathematical Physics. The original motivation for SLE was to study in what way critical statistical physics models are invariant under conformal mappings. Because of conformal invariance, SLEs are the only known continuum limits of interfaces in two-dimensional statistical physics. The rigorous connection between these random interfaces and SLE is the biggest success of the theory. On the other hand although CFT offers a reasonable classification of critical phenomena in 2d, it is a local explanation and it is not geared for study of extended objects such as domain walls. These curves are of great interest as, for example, they describe percolation cluster boundaries, level lines of height models and spin cluster boundaries. Recently, the scaling limit of the spin cluster boundary in the Ising model with appropriate boundary conditions has been proven to be an SLE with diffusivity $\kappa = 3$ [**2**].

An ever increasing number of 2d lattice models have been solved; the techniques have undoubtedly enriched the theory of integrable systems, leading to structures such as the Yang-Baxter equation and quantum groups [**3**]. However it is fair to say that these solutions have added little physical insight into the nature of the critical state.

The striking property of physical systems at criticality is Universality. This means that the physical properties of the system at criticality are independent of the microscopic details. Universality allows one to draw parallels between different systems and at the same time classify critical points. Theoretically, it leads to the conclusion that the behavior near a critical point can be described by just a few relevant parameters such as the dimension and symmetry. It

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turns out that the critical exponents are largely determined by just the dimension of the system, and the dimension and symmetries of the order parameter. This justifies the use of simple models, for investigation of critical behavior, in which all the details of the interactions have been neglected. Such a model as an example of its class will yield all the necessary exponents. For example, the Ising model, the q-state Potts models and the O(n) models are examples of a class of critical behavior. Note that, by changing q or n these models yield various different classes. For instance q = 2 yields the Ising class.

This is an interesting property of these models, for instance in the O(n) model, we can observe the properties of many critical classes for different values of n, sometime even continued to mathematically unacceptable, but physically relevant values such as n = -2 [**4**].

Yet another interesting property of the critical point is that the correlation length at this point diverges, therefore change of scale should not lead to a new theory, hence the emergence of scale invariance. The hypothesis of scale-invariance led to the development of several techniques for the computation of critical exponents and other observables near the critical point [**5**]. In particular one can argue that under relatively general conditions such as conservation of energy and momentum, unitarity and locality, scale invariance leads to conformal invariance [**6**]. This allows the application of the Coulomb Gas method [**4**], a technique which produces exact results for a large class of critical phenomena.

There is an issue remaining, the physicist intuitively believes that there is a connection between the lattice model for example the Ising model on the lattice, and a continuum (the scaling limit) model, in this example the free fermions in 2d. The correspondence is supposed to hold as the lattice spacing tends to zero. A crucial question is, whether this intuition is correct and a mathematical proof can be constructed. Such proofs have proven hard to find. In fact only in two special cases of percolation and the Ising model do we have a proof [**2**]. However good news is that unexpected things have not happened yet and we do not know of any counter examples to this intuition.

The goal of this article is to explain SLE to a reader who is not deeply interested in the mathematical rigor. We give a very brief introduction to SLE and its connection to CFT. We will explain how lattice models are connected with these ideas. For the sake of brevity we do not go into details and the interested reader is referred to original papers and many interesting reviews [**5**,**6**]. There are other statistical models in the picture which are not spin models, such as the self-avoiding walk (SAW), growth models such as the Edwards-Wilkinson model [**7**] and the Kadar-Parisi-Zhang model [**8**]. In fact generally speaking iso-height models also form loop ensembles [**9**] which are similar to the Ising and the O(n) models, and are thus presumably treatable by SLE methods [**10**]. Interestingly certain phenomena such as turbulence and watersheds also yield to the same idea [**11**,**12**], which we shall briefly touch on. Clearly many points are missed or not dealt with to the extent that they deserve; but covering the entire field is only possible in a long review paper or a book, which is planned for a future work.

CONFORMAL INVARIANCE

Before we go any further let us quickly say what a conformal map is and why it is interesting. In Physics symmetry plays an important role, in particular space-time symmetries. The usual space-time symmetry that Physics tends to work with is the Poincare group, consisting of time translations, space translations, rotations in space and boosts. By Noether's theorem these symmetries lead to conserved quantities; correspondingly; energy, momenta and angular momenta and spin. However we can extend these symmetries by two other transformations, dilations and inversions. Dilations require scale invariance of the Physical system and inversions (more often called special conformal transformation) essentially transforms r to 1/r. This extension leads to d+1 extra generators of the symmetry hence the d-dimensional Poincare group is enlarged to SO(d,2). Clearly the existence of scale and special conformal transformations requires very special physical systems. Most systems we know are tied up to a particular set of temporal and spatial scales. It turns out that systems at criticality are scale invariant. This is a direct by product of the Renormalization Group (RG) flow. Under these assumptions of RG, the coupling constants g (which may be a finite set) flow with scale λ:

$$\frac{\lambda dg}{d\lambda} = \beta(g) \tag{1}$$

Where the beta functions β; give the rate of change of the coupling constants. The fixed points of this flow given by $\beta(g^*) = 0$, turn out to be the critical points of the system. In a seminal work Polchinski [**13**] claimed that scale invariance and unitarity lead to conformal invariance, a claim which is very close to a proof now [**14**]. This result has various implications one of which being that critical behavior may be classified by conformal field theories. The restriction of this result to two dimensions brings about a wonderful result. At the outset I must add that, knowledge of two dimensional scale invariance leading to conformal invariance is older and was already known in the paper by Belavin, Polyakov and Zamalodchikov [**15**], where they developed two dimensional conformal field theory (CFT). They observed that two dimensional conformal invariance is very powerful. It is in fact an infinite dimensional symmetry. It turns out that conformal symmetry in 2d is equivalent to all analytic maps of the complex plane to itself. Let D, D′ be domains in complex plane. By definition, a conformal map from D to D′ is an analytic bijection; f: D → D′, such that; $f' \neq 0$ for all $z \in D$, and f^{-1}: D′ → D is also a conformal map. However this definition of a conformal map restricts the possible analytic transformation to the Mobius group:

$$z \rightarrow \frac{az+b}{cz+d'}\; ad - bc = 1 \tag{2}$$

Which is in fact the group SL(2,C). However, giving up the bijection requirement, we can extend the transformations to all analytic function. Therefore if we take the 2d plane as our space-time, the symmetries of the system are extended to all transformations which preserve the angles. This is an infinite dimensional symmetry, hence makes the resulting field theory integrable. There is a second point to consider that is 2d field theories can be continued to the imaginary time, hence 2d CFT may be applied to theories in thermal equilibrium, rather than evolving in time. This makes CFT a very powerful tool for study of critical phenomena. There remains a sneaky obstruction. Quantum field theory is a local theory by construction and is not fit for the study of non-local objects such as domain walls in critical systems. Here comes the power of Schramm-Loewner Evolution, when studying domain walls near criticality, the price to pay is a little abstract mathematics which we hope the reader shows some patience for.

SCHRAMM-LOEWNER EVOLUTION

Consider a path in the upper half complex plane: $\mathbb{R} \rightarrow \mathbb{H}\{z \in \mathbb{C}, Im(z) > 0\}$. We specifically request this path to not intersect, but it can touch itself, see figure. The path is designed such that it starts at origin and stays in the upper half plane for all time: $\{\gamma(0) = 0\ , \gamma(t) \epsilon \mathbb{H}\ \forall t\}$.Loewner [**16**] suggested that there is another way to look at this path. You find the conformal mapping that maps $\mathbb{H}\backslash\gamma$ to $\mathbb{H}$ in such a way that, γ is absorbed into the real line. Then for each t, the domain $\mathbb{H}\backslash\gamma(t)$ is simply connected and by the Riemann mapping theorem [**15**] there exists a conformal map $G(t,z): \mathbb{H}\backslash\gamma(t) \rightarrow \mathbb{H}$. Hence Loewner's claim holds. This map is not totally fixed so we can ask it to have good properties such as; $G(t,0) = 0$, it keeps origin where it was for all time, $G(t,\infty) = \infty$ likewise for infinity, $G'(t,0) = 1$ which is consistent with $G(0,z) = z$ that is at zero time you just get the identity map. We are still left with some freedom expressed as the hydrodynamic limit $G(t,z) \sim z + c/z$ for large z. The constant c is referred to as the capacity, strictly speaking "c" can be a function of time. All this is interesting because Loewner showed that $G(t,z)$ satisfies a differential equation:

$$\partial_t G(t,z) = \frac{2}{G(t,z) - a(t)}, \qquad G(0,z) = z \tag{3}$$

where the real function $a(t)$ is referred to as the "driving function". The normalization of G in Eq. (3) is chosen such that the capacity is c = 2t. The right hand side of Eq. (3) becomes singular at $G(t,z) = a(t)$. Indeed, for each z in the closure of the upper-half plane, such that $G(t,z) = a(t)$, we observe that $G(t,z)$ is real, and simultaneously, $z = \gamma(t)$. Therefore the driving function is the image of the curve $\gamma(t)$:

$$\gamma(t) = G^{-1}(t, a(t)) \tag{4}$$

Thus the mapping G(t,z) maps the tip of the path $\gamma(t)$ to a point on the real axis. Hence you can stick in any continuous driving function and a unique path $\gamma(t)$ emerges. In this way we obtain a trace as depicted in Fig. 1. However a special driving function gives rise to a special Loewner evolution. As pointed out by Schramm [**1**] if you choose $a(t)=\sqrt{k}B(t)$, where $B(t)$ is a standard one dimensional Brownian motion, you get a set of paths in the complex plane which are conformally invariant, this statement needs some clarification. Before doing so let us emphasize that the diffusivity coefficient κ, is a positive real number $0 \leq \kappa \leq 8$, for reasons that become clear below. Now G(t,z) is a random conformal map, it maps a random subset of the upper half plane $\mathbb{H}\backslash K_t$ onto the whole of the upper half plane. The random set K_t is the region of the complex plane excluded by the path (t). In the simplest case ($\kappa \leq 4$) the trace of $\gamma(t)$ does not exclude any region, but as the trace becomes more complex for higher values of κ, the excluded region in the complex plane will become more complex. We shall refer to this excluded region K_t, as the "hull" of $\gamma(t)$. We can rewrite Eq. (3) using $F(t,z) = G(t,z) - \sqrt{k}B(t)$:

$$\partial_t F(t,z) = \frac{2}{F} - \sqrt{k}dB(t) \tag{5}$$

Which is a stochastic differential equation related to the Poisson process [**17**]. This means that the Fokker-Planck equation for this process:

$$\partial_t \psi(F) = \frac{\partial}{\partial F}\left(\frac{2}{F}\Psi(F)\right) + \frac{k}{2}\frac{\partial^2}{\partial F^2}\psi(F) \tag{6}$$

is readily solved for the stationary distribution to get:

$$\psi(F,\infty) \sim F^{\frac{4}{k}} \tag{7}$$

but with clear normalizability issues [**17**]. The probability distribution function $\psi[F]$, if it exists, has to be conformally invariant, which is the crucial property of Schramm-Loewner Evolution (SLE).

The Schramm-Loewner Evolution (SLE) has two important properties, conformal invariance and domain Markov property. Consider a closed simply connected domain $\Omega \neq \mathbb{H}$ such that a path connects two points (a,b) on the boundary of Ω to each other, such that $\gamma(0) = a$, $\gamma(t) = b$. Now consider the conformal mapping $\Phi: \Omega \to \Phi(\Omega) \subset \mathbb{H}$, then we have $\Phi \circ \gamma(0) = \Phi(a)$, under these conditions the conformal measure ψ remains invariant. On the other hand let us break the path at a point c inside the domain Ω, this happens at a time $\tau < t$. Wenow have the domain Ω/K_t which means that the original domain has been polluted with the hull of the path K_t. The natural question is how we deal with K_t. A reasonable answer is that there exists a conformal mapping $\hat{\Phi}: \Omega \setminus K_t \to \Omega$ such that c and b reside on the border of Ω and K_t is completely absorbed into the boundary of Ω. This means that further conditional evolution of SLE path within the original domain Ω happens within Ω/K_t.

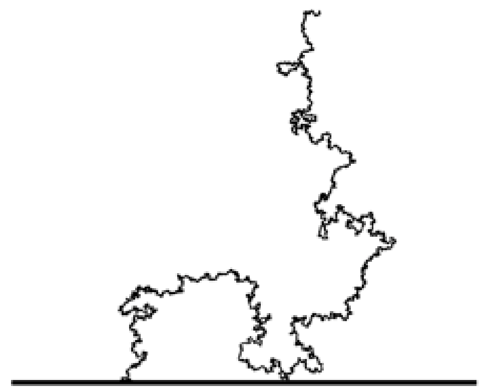

Fig. 1. A non-self intersecting path in the upper half plane.

The Slit Map

Consider a very simple example where the driving function is just a real constant $a(t) = \xi$. Then Eq. (1) is easily solved:

$$G(t,z) = \xi + \sqrt{4t + a(z)\,\xi} \tag{8}$$

Now apply the boundary conditions such as $G(0,z) = z$ to get:

$$G(t,z) = \xi + \sqrt{4t + (z-\xi)^2} \tag{9}$$

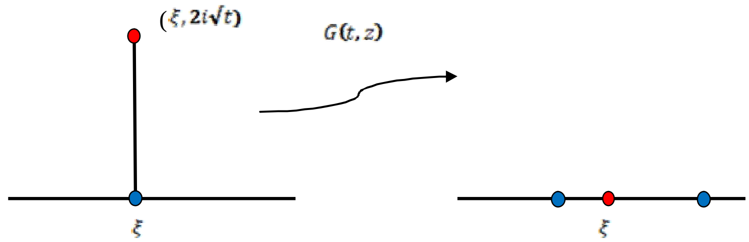

Fig. 2. The slit map. The conformal mapping maps the segment to a point on the real axis. Observe that due to the double valued nature of the slit map the point on the real axis maps into two points on the real axis.

In this case γ(t) is a straight line leading from point ξ on the real line to point (ξ, $2i\sqrt{t}$) on the complex plane, see Fig. 2. The other properties of this function are: $G(0,z) = z$ it maps the tip of the trace (ξ, $2i\sqrt{t}$) to point ξ and other properties for the function listed above. Strictly speaking the slit map is not an SLE but its importance lies in the fact that for very short times, it is an infinitesimal segment which can be used to approximate a bona fide SLE. So repeated application of the slit map can approximate any SLE trace.

Types of SLE

What we have so far introduced here is the chordal SLE. Another type of SLE, called the radial SLE, is a path that starts on the rim of the unit disc U at $z = e^{i\theta}$, and conditioned to end up at z = 0. Then $G(t,z)$: $U \backslash \gamma(t) \rightarrow U$ conformally, such that $G(t,0) = 0$ and $G'(t,0) = e^t$. Loewner's equation in this case is:

$$\frac{\partial G(t,z)}{\partial t} = -G(t,z)\frac{G(t,z)+e^{i\theta}}{G(t,z)-e^{i\theta}}, \qquad G(0,z) = z \tag{10}$$

Here $\theta(t) = \sqrt{k}\, B(t)$, is a scaled Brownian motion. In fact this is the version considered by Loewner [**16**]. At first sight it is not obvious how the radial and chordal versions are related, however it can be shown that if the trace of radial SLE hits the boundary of the unit disc at $t = t_1$, and starts at origin at t = 0, is the same chordal SLE conditioned to begin at origin z = 0 and end up at $e^{i\theta(t)}$, up to a re-parameterization of time. This means that the chordal and radial versions with the same κ, are describing the same physical problem. Multiple radial SLE paths can start at origin and end up on the boundary of the disc. Indeed this is Dyson's procedure for finding eigenvalues of rando unitary matrices [**18**]. The interesting feature of radial SLE is that it can wind around the origin. The winding angle at time t is simply $\theta(t) - \theta(0)$, which is a normally distributed parameter with variance K_t.

Another variant is the dipolar SLE in which the curve is constrained to start at some boundary point and to end up within some finite segment of the boundary not containing the start point [**19**].

Another variant of SLE named SLE(κ,ρ) [**20**,**21**], is obtained by changing the driving function from the Brownian motion to W_t given by:

$$dW_t = \sqrt{k}dB_t + \frac{\rho}{W_t - G_t}dt, \tag{11}$$

where the iterated conformal mapping G_t is given by Loewner's equation:

$$dG_t = \frac{2}{G_t - W_t}dt \tag{12}$$

This process is well defined for κ > 0, ρ > -2, and the inequality $W_t - G_t > 0$ is valid for all positive time. Obviously for ρ = 0 we recover the original SLE. The restriction property set above means that while the SLE trace emanates from origin, there is a preferred point on the real axis, on the negative real axis in this case which corresponds to W_0. Such a prefered point can be constructed in the spin models by inserting a boundary condition changing (bcc) operator [**5**] .

As we have a generalization of CFT to logarithmic CFT [**22**], a generalization of SLE to logarithmic SLE is also possible [**23**,**24**], where a couple of traces are generated. An application of this idea has yet to be found.

SOME PROPERTIES OF SLE

Here we give some of the properties of SLE, the list is not exhaustive. The interested reader should refer to the original papers for a more complete listing.

Continuity

An obvious question arises as to whether the trace obtained by the inverse mapping given in Eq.(4) is a continuous path. The proof is given in [**25**] and completed in [**21**]. In other words the following theorem holds: For all $\kappa \geq 0$ the limit $\gamma(t) = \lim_{z\to 0} G^{-1}(z+\sqrt{k}\, B(t))$ exists.

Transience

The other question that arises is whether the trace always ends up in infinity or not? The answer is affirmative and was proven in [**25**].

Phases of SLE

As already mentioned the SLE trace has three distinct phases. For $4 > \kappa \geq 0$ the SLE trace is a simple path never touching itself and rising up to infinity. For $8 > \kappa \geq 4$ the trace is a curve that can touch itself and the real axis, eventually ending up at infinity. This means that there exists an area, K_t, separated by the trace from infinity, called the "Hull". For $\kappa \geq 8$ the path is space filling.

Fractal Dimension

It is clear that the SLE trace is a fractal embedded in the complex plane; so we could ask what the fractal dimension of this path is. This has been calculated by Beffara [**26**]:

$$d_f = 1 + \frac{k}{8} \tag{13}$$

The three phases of SLE corresponds to interesting fractal dimensions. For $4 > \kappa \geq 0$, ($d_f < 3/2$). For $8 > \kappa \geq 4$ ($3/2 \leq d_f < 2$), and the trace becomes space filling for $\kappa = 8$, $d_f = 2$, and equation (11) does not hold for $\kappa > 8$, since it implies a dimension greater than 2 for a geometrical object embedded in 2d which is impossible.

Duality

Another interesting property is SLE duality. Let us first define the hull of an SLE. The SLE trace, touching itself and the real axis may exclude regions which are not on the path but nevertheless are not reachable from infinity. We call the union of the set of such point together with the curve itself, up to time t, the hull K_t. For large $\kappa < 8$, the curve tends towards being space filling, so it continually touches itself. Hence the hull contains all earlier parts of the path. However, the boundary of K_t, *i.e.* the boundary of $\mathbb{H} \backslash K_t$ minus any portions of the real axis, is a simple curve. Thus this frontier or boundaries by definition forms a simple curve, proven to be another SLE but with diffusivity κ', such that $\kappa\kappa' = 16$ [**27**,**28**]. Thus the self-dual point is $\kappa = 4$, corresponding to the iso-height lines of a Gaussian Free Field (GFF) or the Edwards-Wilkinson model [**7**], [**29**]. Later in this review we shall see that the two dual SLE's have the same central charge.

Locality

For $\kappa < 4$ we clearly have a self-avoiding trace, but clearly not a self-avoiding random walk (SAW). What is happening is that the SLE trace exerts a force on itself and the boundary of the domain. At first sight this seems a mysterious property, but it actually results from the non-locality of the SLE process. However there is a special value of κ for which locality exists, *i.e.* the SLE trace evolves independent of its boundary and history so long as it does not cross itself. This special value is $\kappa = 6$ [**30**]. In other words critical percolation is represented by a local SLE.

Restriction

The acute reader would point out that if percolation has a locality property then it's dual *i.e.* $\kappa = 8/3$, which corresponds to SAW should have a special property too, though not locality. In fact the trace of $SLE_{8/3}$ is the covering of the hull of SLE_6. To explain, sacrificing mathematical exactitude; consider the mapping which maps $\mathbb{H} \backslash K$ to $\mathbb{H}$:

$$G_K(z): \mathbb{H}\backslash K \to \mathbb{H} \tag{14}$$

This is in fact what a Loewner mapping does to the Hull of an SLE trace for $\kappa > 4$, in particular also true for $\kappa = 6$. In this section I shall restrict myself to saying K only for $\kappa = 6$.

Now define a new mapping $\Phi(z) = G_K(z) - G_K(0)$. This mapping clearly maps the upper half plane $\mathbb{H}$ \K onto itself such that $\Phi(0) = 0$. Now if the trace $\gamma(t)$ never hits K, then so is true for $\gamma^*(t) = \Phi(\gamma(t))$. In other words γ, γ^* have the same distribution. It can further be shown that [**31**]:

$$P(\gamma \cap K = \emptyset) = |\Phi(0)|^{5/8} \tag{15}$$

The two properties of locality at $\kappa = 6$ and restriction at $\kappa = 8/3$ can be stated as below, sacrificing mathematical exactitude. Let U be a subset of $\mathbb{H}$, connected to the x axis, such that the trace $\gamma(t)$ starts outside U. The *locality* property ($\kappa = 6$) means that the probability ensemble of $\gamma(t)$ is indifferent to U. So if U is removed by a conformal mapping the probability of $\gamma(t)$ will remain untouched. The restriction property ($\kappa = 8/3$) means that if $\gamma(t)$ does not hit U, it will not hit U in any conformal mapping of $\mathbb{H}$.

CONNECTION WITH CONFORMAL FIELD THEORY

At criticality, we expect a physical system to have conformal invariance. This seems to be a very deep property of physical systems that at scale invariant points (given other properties such as unitarity and Poincare invariance) they become conformally invariant [**32**]. In two dimensions this property is particularly potent since 2d Conformal Field Theory (CFT) is solvable [**15**]. The interested reader may consult many well written reviews and books on this subject, for example [**33**]. Here I wish to avoid reviewing CFT for the sake of brevity. Instead I shall use what is needed from CFT economically. The power of CFT is in the fact that it infers things on the theory just from infrastructure without the need to cite a particular Hamiltonian or Lagrangian. This is done by looking at the irreducible representations of the Virasoro algebra, namely a highest weight state which generates a representation.

In two dimensions, conformal transformations of the plane translate into holomorphic transformations of the complex plane. This is an infinite dimensional symmetry group and results in an affine algebra of sl(2,C), namely the Virasoro algebra:

$$L_n, L_m] = (n-m)l_{n+m} + \frac{c}{12}n(n^2-1)\delta_{n+m}, \qquad n,m \in Z \tag{16}$$

The real positive number c, is called the central charge and is related to the conformal anomaly. There is a similar algebra for the anti-holomorphic operators acting on the complex conjugates. Hence we have a chiral algebra. Unitarity requires $\bar{c} = c$. The standard method for building representations of the Virasoro algebra is similar to the way physicists build representations of the rotation group, *i.e.* by highest weight vectors.

Such a state in CFT is called a Primary field. The conformal weight of a primary field is solely determined by the central charge c. Thus different statistical models are classified by different possible values of the central charge.

$$c = 1 - \frac{6}{(m(m+1)}, \qquad m = 2,... \tag{17}$$

Some of the corresponding critical statistical mechanics models are given in Table 1. The interesting range of the central charge is $0 \le c \le 1$, corresponding to minimal models, which are CFTs having a finite number of primary fields. Clearly both SLE and CFT are describing the same physical system at criticality hence they must be connected. In relation to SLE, we can observe that the Fokker-Planck equation associated with SLE (Eq. 6), is a level two null vector in a Conformal Field Theory. This observation implies that we have a primary operator $\varphi_{2,1}$ and central charge:

$$c = \frac{(3k-8)(6-k)}{2k} \tag{18}$$

Rational CFTs correspond to central charge being less than 1 and rational. These CFTs have a finite number of primary fields. Thus different values of diffusivity coefficient

Table 1. Different Critical Models and their CFT Equivalents

M	c	Statistical model
3	1/2	Ising
4	7/10	Tricritical Ising
5	4/5	3-State Potts
6	6/7	Tricritical 3-state Potts

corresponds to different CFT models, hence to different critical points. Note that the central charge vanishes at κ = 6 (percolation) or κ = 8/3 (the self-avoiding walk). Corresponding to the limit of q = 1 in the Potts model and n = 0 in the O(n) model respectively. Thus there is a connection between states of the Hilbert space of the Boundary Conformal field Theory (BCFT) and the trace of SLE [**34**].

The central charge can be re-written as:

$$c = 13 - 6(\frac{k}{4} + \frac{4}{k}) \qquad (19)$$

Indicating that there is a self-dual point at κ = 4, c = 1, and there is duality between points above and below it.

On the boundary we have a bcc operator, which changes the boundary condition at the point where the SLE trace emanates out. Also on the lattice a bcc operator guarantees the passage of a critical boundary. Thus a trace conditioned to begin at origin and end point r, is related to the green function:

$$\langle \varphi_{2,1}(0)\varphi_{2,1}(r)\rangle \sim \frac{a}{|r|^{2h}} \qquad (20)$$

Here h = $h_{2,1}$:

$$h_{2,1} = \frac{6-k}{2k} \qquad (21)$$

Such a result is easy to get in CFT but extremely hard in SLE.

I close this section by stating a dilemma. The CFT models referred to here, correspond to rational values of κ, whereas no such constraint from SLE side is necessary. What is the root of this constraint from the SLE side, or is it simply spurious? These are questions yet unanswered.

LATTICE MODELS

Let us now look at lattice models in statistical Physics, which at criticality are believed to be related to CFT's. Near the critical model the correlation diverges, hence the lattice model may be well described by a continuous theory, in this case a CFT. This limit, sometimes referred to as the scaling limit equally corresponds to the limit at which the lattice spacing vanishes. In other words the lattice parameter relative to the correlation length may be ignored and the physical system appears to be a continuum. Hence a smooth SLE curve can approximate a random path on the lattice. The random path taken on the lattice may be the path dividing different domains (a domain wall) or simply a self-avoiding random path. It therefore becomes a difficult mathematical task to prove that the scaling limit of a lattice model exists and corresponds to a particular SLE. That such a limit exists has been only rigorously proven in two cases, the Ising model and percolation [**2**]. However there is an expectation that all the statistical models listed in Table 2 (and many more) do have a scaling limit. In the scaling limit critical phenomena are especially interesting, since they exhibit universality. Universality means that many different physical models which differ in their microscopic details, have the same scaling limit. Thus a very important question is to determine what the possible scaling limits are.

Essentially what we are aiming at are conformally invariant random curves in two dimensions. I shall concentrate on a square lattice but in certain cases a honeycomb lattice proves to be easier to deal with. A lattice domain D is a domain in the usual sense, which can be decomposed as a disjoint union of open sets (the usual sets are either triangles, squares or hexagons) with area "a^2" (faces, the area of a face depends on the shape we choose but dimensionally it is a^2), open segments of length "a" (edges or bonds) and points (vertices), in such a way that each bond belongs to the boundary of two open sets and each vertex belongs to the boundary of "n" open edges, again n depends on the choice of unit cell in the lattice. A path on D is a sequence {s_1…s_n} of bonds starting at a vertex s_1, and ending on a vertex s_n. Usually there is no break in the sequence, *i.e.* we have a connected path. Because arbitrary domains can be used, statement of conformal Invariance is by no means trivial. Specifically we are looking for probability measures on these domains, and sequences of domains while the meshing "a" tends to zero in such a way that the limit exists, is conformally invariant and obeys domain Markov property. Only in this way can we say that an SLE corresponding to the scaling limit of a lattice model has been found.

In Physics other criteria are added, for example a spin

Table 2. The Values of Diffusivity, Central Charge and Fractal Dimension for Various Critical Models in Statistical Physics

κ	2	8/3	3	10/3	4	24/5	16/3	6	8
Central charge	-2	0	1/2	4/5	1	4/5	1/2	0	-2
q	4	0	1	2.63	4	3	2	1	0
n	-2	0	1	1.62	2	1.73	1.41	1	0
Fractal dim.	5/4	4/3	11/8	17/12	3/2	8/5	5/3	7/4	2
Model	LERW ASM	KPZ SAW	Ising Spin Cluster	3-State Potts	Edwards-Wilkinson GFF Kosterlitz-Thouless	Dual 3-State Potts	Ising FK Clusters	Percolation Turbulence	Universal Spanning Trees

system is defined on the square lattice, and the spins on one side of the sequence are positive, whilst they are negative on the other side of the sequence. Hence a domain boundary is formed. It therefore becomes possible to assign a Boltzmann weight W_p to the configuration **p**, and the partition function is determined by the sum:

$$Z = \sum_p W_p \tag{22}$$

If we have a path p, running from r_1 to r_2, it is tempting to associate this path with a Green function. So let $W_p^{1,2}$ are the Boltzmann weights associated with all the configurations with all sequences running from r_1 to r_2, then:

$$G(1,2) = \frac{\sum_p W_p^{\;1,2}}{\sum_p W_p} = \langle \varphi_h(r1)\varphi_h(r2)\rangle \tag{23}$$

The operator φ_h is chosen suitably to create the start and end points. The best understood model is the Ising model, which at criticality is related to a c = 1/2, CFT. Hence it is expected that domain walls at the critical point of an Ising model should correspond to SLE traces with κ = 3.

The Ising Model

Consider an Ising model realized on a honeycomb lattice in 2d. (See Fig. 3). At the center of each hexagon we have placed a spin which can take on values +1 or -1. The indices i and j uniquely determine the position of the hexagon on the plane. The Ising Hamiltonian is given by:

$$H = -J\sum_{\langle ij,kl\rangle} S_{ij}S_{kl} + h\sum S_{ij} \tag{24}$$

The sum is over all neighboring spins, the symbol <*ij,kl*> means sites which are neighbours to each other, here six for the hexagon. There is an applied magnetic field **h**, as well.

Interpret the bottom of the lattice as the real x axis, and here we impose a change of boundary condition using a bcc operator. This forces the start of the SLE path on the real axis (or origin). The path then will go up keeping positive spins on the left and negative spins on the right. Alternatively this is truly a domain wall on the dual triangular lattice. The SLE trace is the continuum limit of such a trace at the critical coupling of the Ising model.

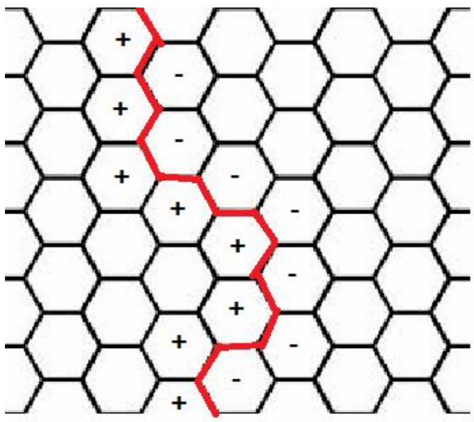

Fig. 3. Ising model on a Honeycomb lattice, with an SLE trace running from the bottom to the top edge, separating positive and negative spins.

The loop picture of the Ising model is interesting here. Consider the partition function in the limit of vanishing magnetic field:

$$Z = \sum e^{\beta J \sum_{\langle ij,kl \rangle} S_{ij} S_{kl}} \tag{25}$$

This can be expanded in terms of products of spins:

$$Z = Tr \prod_{S_{ij}} (1 + x\, S_{ij} S_{kl}) \tag{26}$$

Here = tanh (βJ). The trace of products vanishes unless we have a closed loop, so the partition function can be written as a sum over loops:

$$Z = \sum_{g} x^{L(g)} \tag{27}$$

The sum is over all graphs g, on the Ising lattice which has loops, L(g) is the total length of the graph. The loops in g are domain walls separating up and down spins. At high temperatures β is small, thus x is small, and hence, the mean length of a loop is small. As expected in high temperatures we do not have large domains. However for small temperatures the mean loop perimeter is large, as large domains of spins form and magnetization is nonzero. These two phases are separated at x_c where the loop perimeter diverges. Therefore at x_c, we expect the loop perimeters to correspond to SLE traces. This model corresponds to κ = 3.

The O(n) Model

The O(n) model is a generalization of the Ising model, which has a loop expansion as well. The other interesting property of the O(n) model is that it corresponds to many critical models at various values of n. The Hamiltonian of the O(n) model is given in terms of an n-component vector $S^a(ij)$, $a = 1 \ldots n$. The indices i and j refer to the position of the spin on the complex plane. The spins are orthonormal:

$$Tr(S^a(ij) S^b(ij)) = n\delta^{a,b} \tag{28}$$

No sum over the position identifiers ij. Hence the partition function of loops again arises:

$$\boldsymbol{Z \equiv Tr \prod_{S_{ij}} (1 + x\vec{S}_{ij} \cdot \vec{S}_{kl})} \tag{29}$$

Then we have:

$$Z = \sum_{g} x^{L(g)} n^{c(g)} \tag{30}$$

Where c(g) is the number of connected components of g. At n = 1 we recover the Ising model, and for other values of n, just like the Ising model, the loop perimeter diverges above some n-dependent value, $x_c(n)$, *i.e.* we have critical behavior above $x_c(n)$. Some interesting values of n are: n = 2, where we have the XY-model. In this case spins can be related to heights over a lattice and we obtain the (GFF), where κ = 4 (see Table 2). The case n = 0 corresponds to the Self Avoiding Random Walk (SAW) with κ = 8/3, and the case n = -2 is the loop erased random walk (LERW) with κ = 2. In general the interesting cases happen between n = -2 and n = 2. There is a relation with κ:

$$\boldsymbol{n = -2Cos(\frac{4\pi}{k}), \qquad k \le 4} \tag{31}$$

Note that if we set n = $\sqrt{q}$, we observe a relation with the q-state Potts model:

$$\boldsymbol{q = 4Cos^2(\frac{4\pi}{k}), \qquad 4 \le k \le 8} \tag{32}$$

There are two phases in the critical O(n) models, the dilute and the dense phase [**35**]. This refers to the density of loops on the lattice. The two phases correspond to the two phases of SLE. So in Eq. (31), we can replace $\kappa/4$ by $4/\kappa$. Therefore the dilute phase holds for $\kappa < 4$ while the dense phase holds for $\kappa > 4$. There is no O(n) model corresponding to $0 < \kappa < 2$ [**6**], this observation is of relevance to the watershed case discussed at the end of this article [**12**].

Potts Model

The q-state Potts model is a lattice spin model where the spins s(r) take values from the set {1, 2..., q} and have nearest neighbor interactions, such that the partition function is given by:

$$Z = Tr(e^{\beta J \sum_{<R,R>} \delta_{S(r)S(r)}}) \tag{33}$$

The case q = 2 is equivalent to the Ising model and q = 1 corresponds to percolation, and for two other integer values of q it displays a continuous phase transition: q = 3 and q = 4. For q > 4 the transition is first order. Therefore, the question arises if spin boundaries in the critical Potts model with other values of q have SLE as their scaling limit or not. The answer is affirmative [**36**]. The 3-state Potts model has spin boundaries that correspond to $\kappa = 10/3$ (see Table 2). There is an obvious difficulty in defining a spin boundary, for a three state system, it is no longer unique as it was in the Ising spin system. The answer comes in looking at the boundary of two spins against the third one. However this choice forces fluctuating boundary conditions, which should be handled correctly [**36**].

PERCOLATION

In general, percolation refers to the flow of fluids through porous media. However an abstract mathematical formulation of this problem is in mind here. Consider a lattice on the complex plane whose unit cell is a hexagon. Color each hexagon randomly white or black. Set the probability of being white as p. Ask at what "p" do I see a globally connected cluster connecting the bottom and top of the lattice. For infinitely large lattice, this defines the critical percolation problem. The answer in this case is p = 1/2 and has a simple proof by Kesten [**37**].

This model exhibits a continuous phase transition with its own characteristic exponents, the critical behavior of percolation is such that the boundary is an SLE trace $\kappa = 6$. The connection between Percolation and SLE_6 was proven by Smirnov [**2**].

What makes percolation even more interesting is that it corresponds to a c = 0 CFT, *i.e*. a non-unitary theory with logarithmic correlators, and perhaps intimately related to disorder. The other peculiar property of percolation is that SLE_6 is local, *i.e.* not sensitive to the boundaries. An interesting consequence of the percolation-SLE connection was the proof of a conjectured crossing formula for the probability that in critical percolation a cluster should exist which spans between two disjoint segments of the boundary of some simply connected region on the complex plane [**38**], giving ground to the use of local operators from CFT in percolation theory.

The dual of percolation is the Self Avoiding Random Walk (SAW), dual to the locality property, SAW exhibits restriction property. This property of SLE happens only at $\kappa = 8/3$, as explained before. Note that the two values of central charge at c = 0 and c = -2 are non-unitary, however they are related to very important models, Percolation and the Abelian Sandpile Model (ASM).

SURFACE GROWTH AND ISO-HEIGHT CONTOURS

There is another way that we can end up with a random ensemble of loops, by looking at the iso-height lines of a random height surface. So given a height profile over a 2d surface h(x), let us look at the contours h(x) = constant (here I shall use x to mean a two component vector). This will create a random collection of loops, somewhat like the loop models such as the Ising model or the O(n) model. The question now is can I find a probability measure over such an ensemble. There is a formal answer to this question. Given a probability distribution for h $P[h]$ we can form the probability distribution of h = *Constant*; $P_c[h]$ readily by the following path Integral:

$$P_c[h] = \int [dh] P[h] \delta[h-c] \tag{34}$$

A popular case is that of Gaussian distribution for the height:

$$P[h] \approx \exp\{-\beta \int d^2k\, k^{2\alpha+2} |h(k)|^2\} \tag{35}$$

Where h(k) is the Fourier transform of h(x). Equation (26) implies a scaling invariance for h:

$$h(x) \cong \lambda^{\alpha} h(x) \tag{36}$$

The symbol ≅ in Eq. (36) should be understood as -has the same probability.

Identifying α as the Hurst index of the rough surface h(x):

$$\langle h(x)h(y)\rangle \approx |x-y|^{2\alpha} \tag{37}$$

The obvious example is the GFF, with κ = 0, corresponding to O(2) or the Edwards-Wilkinson model of surface growth [**39**]. The equal height contours correspond to CLE_4 [**29**,**10**]. What has to be clarified here is that actually loop ensembles are related to Conformal Loop Ensembles [**40**]. However the mathematical difficulty of these constructs has hindered their application to physical systems, therefore many authors have used SLE instead. The study of rough surfaces using their equal height contours started with [**9**]. The surface defined by h(x) is a fractal embedded in three dimensions which fractal dimension 3-α. The fractal dimension of the contour line, for a Gaussian surface (Eq. 35) is 3-α/2 [**9**]. The set of lines, perpendicular to the contours, in a sense the route a drop of rain would take rolling down the hill, is not known, but known to be smaller than 3-α/2 [**12**]. Surfaces other than GFF have been studies but results are not conclusive yet. For instance the Kardar-Parisi-Zhang (KPZ) surface seems to belong to the SAW class [**10**]. In the same way one can study actual rough surfaces grown in the laboratory. These surfaces are not always self-affine, but when they are, they can be good candidates for having SLE traces as iso-height contours [**41**]. Equal height surfaces are not the only candidates for creating loop ensembles. Another interesting candidate is the ensemble created by looking at constant vorticity contours in 2d turbulence [**11**].

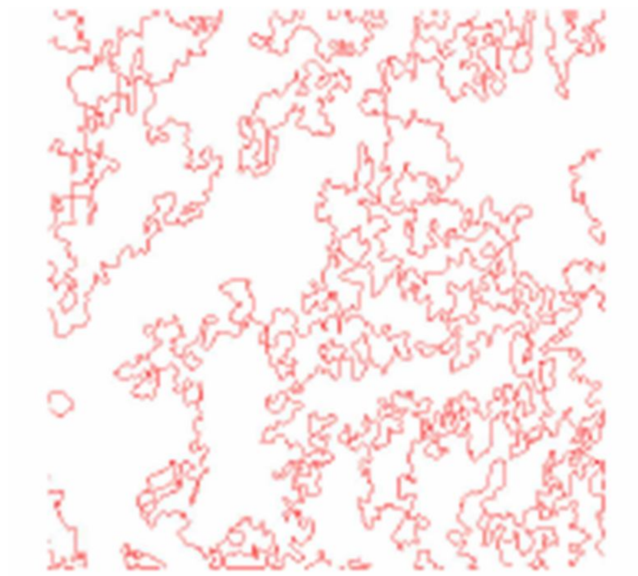

Fig. 4. Equal height contours of WO_3 film on glass, obtained by Atomic force microscope imaging. The contours correspond to SLE_3 [41].

WATERSHEDS

Watersheds are lines on a topography separating adjacent drainage basins. So they are the lines of maximum descent and ascent connecting adjacent saddle points and peaks, in such a way that a collection of local minima is encompassed, so that the overflow of each local minimum pours into another local minimum within the same set. Watersheds play a fundamental role in water management, landslides, and wars. Natural watersheds are fractal [**42**]. Given a landscape, either artificial or real, finding the watershed line requires the running of an invasive percolation algorithm. It therefore transpires that watersheds are indeed the cluster boundaries of invasion percolation. Invasion percolation first discussed in [**43**] was motivated by the study of the flow of two immiscible fluids in porous media. The mathematical definition of the problem is based on a 2d lattice Let us assign a random number, drawn from a given distribution on the unit interval [0, 1], to each link in the lattice. In initial configuration, the defender forms a spanning cluster, while the invader resides entirely in some compact region. Starting from such an initial configuration, the invader grows at each time step by displacing the defender along that link on the interface which has the smallest value. The boundary forms a fractal with fractal dimension depending on the probability distribution of the defending links. For a Gaussian d_f = 1.22. Using this method, the fractal dimension was estimated to be 1.10 ± 0.01 for the Alps and 1.11 ± 0.01 for the Himalayas. Indicating a dependence on the Hurst index [**44**]. Furthermore the universality class seems to be related to paths in random media [**45**]. Armed with the machinery of SLE, we can observe that in the continuum limit watersheds are Schramm-Loewner Evolution (SLE) curves, with κ = 1.734 ± 0.005 [**12**]. This value of diffusivity lies outside the observed range ($2 \le \kappa \le 8$) but not necessarily inconsistent with the SLE property. It can have two meanings; either relation with CFT does not hold or a logarithmic CFT with central charge of c = -3.47 ± 0.05 is playing a role.

SUMMARY

The main goal of this article was to provide a review of recent activity in the Schramm Loewner Evolution (SLE) and its connections to a variety of models in probability theory and statistical Physics. SLE is a one parameter family of stochastic processes that produce non-intersecting random curves in the plane, in such a way that the probability measure for these curves is invariant under the analytic maps of the complex plane and has domain Markov property. It was introduced by Oded Schramm in a paper that appeared in year 2000. He named it Stochastic Loewner Evolution, but now it is referred to as Schramm-Loewner Evolution.

Although the theory of SLE is important in its own right within the domain of Mathematics where a proof of the continuum limits of lattice models is studied [**2**], the Importance of SLE arises out of it as a tool to study non-local entities such as domain walls in physical systems near criticality. In two and four dimensions, it is now believed that critical phenomena are conformally invariant. In other words near a critical point the trace of the energy-momentum tensor vanishes:

$$T^{\mu}_{\mu} \approx \beta(g) \tag{38}$$

Where T^{μ}_{ν} is the energy-momentum tensor and $\beta(g)$ is the beta function, giving the rate of change of the coupling constant, under scaling. This makes the theory scale invariant, hence conformally invariant if some other properties hold [**14**]. Note that the right hand side of Eq. (38) has to be understood in a loose way as there may be many coupling constant. However CFT is a local theory and unfit for study of non-local entities such as domain walls. However SLE is related to CFT *via* its Fokker-Planck equation and offers a way of studying domain walls in conformally invariant systems. Perhaps the most striking success of SLE was in percolation theory, Cardy's crossing formula could be checked [**38**] [**5**], and more recently some problems of paths on critical percolation cluster were addressed [**46**] [**47**]. Certain other problems have also been addressed by SLE such as; boundaries of zero vorticity [**11**], spin cluster boundaries in a number of models for example three state Potts model [**36**], fractal dimension of watersheds [**12**], random surface structure in Liouville quantum gravity [**48**], classification of random surfaces using their iso-height lines [**10**], this is citing just a fraction of the results.

However some problems remain, which can be the focus of future work. Domain boundaries are typically loops, whereas SLE addresses a path starting from a specific point. Making matters more intense, many two dimensional models are really loop models. Meaning that the Boltzmann weights distinguish configuration of loop soups, for instance in the O(n) model [**4**]. This means that we should directly treat loops rather than paths. A mathematical theory, Conformal Loop Ensembles (CLE) dealing with loops has been developed [**49**], but its non-local structure has made it difficult for application to physical problems, this is indeed an avenue of future development. Another direction for future work may be offered by systems near criticality, hence an off-critical SLE becomes important [**50**] [**51**]. However the theory of off critical SLE needs some more development and more interesting results will come out in this field.

Finally, there exist other growth problems which can be generated by iterated analytical complex maps. Another well-known problem of this type is the diffusion limited aggregation (DLA) [**52**]. The resultant highly branched structures are very similar to those observed in viscous fingering experiments [**53**] where one fluid is forced into another in which it is immiscible. Hastings and Levitov [**54**] have proposed an approach to this problem using iterated conformal mappings. It is an interesting as yet open question does there exist a stochastic differential equation for generating these kinds of growth.

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